Advanced Studies in Contemporary Mathematics
30 (2020), No. 1, pp. 121 - 134

www.jangjeon.or.kr
http://dx.doi.org/10.17777/ascm2020.30.1.121

A DIFFERENCE OF SUMS OF FINITE PRODUCTS OF LUCAS-BALANCING POLYNOMIALS

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ABSTRACT. The Lucas-balancing numbers arise naturally from the balancing numbers which were introduced by Behera and Panda about twenty years ago and have been undergone intensive studies by many researchers. Natural extensions of the Lucas-balancing numbers are the Lucas-balancing polynomials. In this paper, we will consider a difference of sums of finite products of Lucas-balancing polynomials and represent them in terms of nine orthogonal polynomials in two different ways each. In particular, this gives us an expression of such a difference of sums of finite products in terms of Lucas-balancing polynomials. Our proof is based on the recent observation by Frontczak as to a fundamental relation between Chebyshev polynomials of the first kind and Lucas-balancing polynomials.

1. INTRODUCTION

In this paper, we consider the following difference of sums of finite products of Lucas-balancing polynomials $C_n(x)$ given by

$$\begin{aligned} &\sum_{l=0}^{n} \sum_{i_1+\cdots+i_{r+1}=n-l} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x) \\ &-\sum_{l=0}^{n-2} \sum_{i_1+\cdots+i_{r+1}=n-l-2} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x), \quad (n \geq 2, r \geq 1), \end{aligned} \tag{1}$$

and represent them in terms of nine orthogonal polynomials in two different ways each. In particular, this gives a representation of (1) in term of Lucas-balancing polynomials.

The Lucas-balancing numbers C_n arise naturally from the balancing numbers B_n, which were introduced by Behera and Panda [3] about twenty years ago. Since their introduction, the balancing numbers have been undergone intensive studies and lots of interesting results about them have been uncovered [4, 7, 11-15]. Natural extensions of balancing numbers and those of Lucas-balancing numbers are respectively the balancing polynomials $B_n(x)$ and the Lucas-balancing polynomials $C_n(x)$. Then $B_n(x) = B_n(1)$, $C_n = C_n(1)$, $(n \geq 0)$.

Our representations of the difference in (1) are based on the fundamental relations between Chebyshev polynomials of the first kind and Lucas-balancing polynomials which was recently discovered by Frontczak in [5]. Similar relations exist between Chebyshev polynomials of the second kind and balancing polynomials. It is very strange that these relations were observed only very recently. From these fundamental observations together with some well-known properties of Chebyshev polynomials of the first kind, we can obtain not only the above-mentioned representations but also some immediate results for Lucas-balancing polynomials and hence also for Luca-balancing

2010 *Mathematics Subject Classification.* 05A19; 33C05; 33C45.

Key words and phrases. Lucas-balancing polynomials; Chebyshev polynomials of the first kind; difference of sums of finite products; orthogonal polynomials.

numbers.

The classical linearization problem in general consists in determining the coefficients in the expansion of the product of two polynomials in terms of an arbitrary polynomial sequence. Thus our representation of the difference in (1) may be viewed as a generalization of this classical problem. In recent years, many mathematicians have drawn their attention on the problem of representing sums of finite products of some special polynomials in terms of other special polynomials [8,9 and the references therein].

We will spend the rest of this section in recalling some preliminary facts and results on the Lucas-balancing numbers and polynomials and on the balancing numbers, and Chebyshev polynomials of first kind. We recommend the Ph. D. thesis of Ray [14] and the survey paper [11], for more details on the balancing and Lucas-balancing polynomials and numbers. As to a general reference for Chebyshev polynomials, we let the reader refer to [10].

The Lucas-balancing polynomials $C_n(x)$ are given either by the generating function

$$\frac{1-3xt}{1-6xt+t^2}=\sum_{n=0}^{\infty}C_n(x)t^n, \tag{2}$$

or by the recurrence relation :

$$C_{n+1}(x)=6xC_n(x)-C_{n-1}(x),\ (n\geq 1),\ C_0(x)=1,\ C_1(x)=3x. \tag{3}$$

The first few terms of Lucas-balancing polynomials are given as follows :

$$C_2(x)=18x^2-1,\quad C_3(x)=108x^3-9x,\quad C_4(x)=648x^4-72x^2+1,$$

$$C_4(x)=3888x^5-540x^3+15x,\cdots.$$

The $C_n=C_n(1)$, $(n\geq 0)$, are called the Lucas-balancing numbers. On the other hand, the balancing numbers B_n are given either by the generating function

$$\frac{1}{1-6t+t^2}=\sum_{n=0}^{\infty}B_nt^n, \tag{4}$$

or by the recurrence relation

$$B_{n+1}=6B_n-B_{n-1},\ (n\geq 1),\ B_0=1,\ B_1=6. \tag{5}$$

We remark here that B_0 is defined as 1, not as 0, following the original definition in [3].

The balancing numbers B_n are originally defined as follows : A positive integer n is called a balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

holds for some positive integer r, in which case r is the balancer corresponding to the balancing number n. From this fact, it is easy to see that a positive integer n is a balancing number if and only if $8n^2+1$ is a perfect square. Then, $c_{n+1}=\sqrt{8B_n^2+1}$, $(n\geq 0)$, are the Lucas-balancing numbers.

Next, we recall a few facts on the Chebyshev polynomials of the first kind and state the corresponding ones for the Lucas-balancing polynomials that follow from aforementioned discovery of Frontczak.

The Chebyshev polynomials of the first kind $T_n(x)$ are given either by the generating function

$$\frac{1-xt}{1-2xt+t^2}=\sum_{n=0}^{\infty}T_n(x)t^n, \tag{6}$$

or by the recurrence relation

$$T_{n+1}(x)=2xT_n(x)-T_{n-1}(x),\ (n\geq 1),\quad T_0(x)=1,\quad T_1(x)=x. \tag{7}$$

In addition, they are explicitly given by

$$T_n(x)={}_2F_1\left(-n,\ n\ ;\ \frac{1}{2}\ ;\frac{1-x}{2}\right)=\frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(2x)^{n-2l},\quad (n\geq 1), \tag{8}$$

where ${}_2F_1(a,b;c;x)$ is the Gaussinan hypergeometric function, (see (24)).

The next fundamental result, first observed in [5], follows either (2) and (6), or (3) and (7).

Lemma 1.1. ([5]) *For any integer $n\geq 0$, we have*

$$C_n(x)=T(3x). \tag{9}$$

As is knwon, $T_n(x)$ satisfies the orthogonality relation

$$\int_{-1}^{1}(1-x^2)^{-\frac{1}{2}}T_n(x)T_m(x)dx=\frac{\pi}{\varepsilon_n}\delta_{n,m}. \tag{10}$$

From (10) and (9), we can derive the following orthogonality relation of $C_n(x)$:

$$\int_{-\frac{1}{3}}^{\frac{1}{3}}(1-9x^2)^{-\frac{1}{2}}C_n(x)C_m(x)dx=\frac{\pi}{3\varepsilon_n}\delta_{n,m}. \tag{11}$$

Moreover, from the Rodrigues' formula for $T_n(x)$ given by

$$T_n(x)=\frac{(-1)^n2^nn!}{(2n)!}(1-x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-x^2)^{n-\frac{1}{2}}, \tag{12}$$

we obtain

$$C_n(x)=\frac{(-1)^n(\frac{2}{3})^nn!}{(2n)!}(1-9x^2)^{\frac{1}{2}}\frac{d^n}{dx^n}(1-9x^2)^{n-\frac{1}{2}}, \tag{13}$$

Now, we get the following proposition from (11) and (13).

Proposition 1.2. *Let $q(x)\in\mathbb{R}[x]$ be a polynomial of degree n, and let $q(x)=\sum_{k=0}^{n}\beta_kC_k(x)$. Then we have the following:*

$$\beta_k=\frac{3\varepsilon_k(-\frac{2}{3})^kk!}{\pi(2k)!}\int_{-\frac{1}{3}}^{\frac{1}{3}}q(x)\frac{d^k}{dx^k}(1-9x^2)^{k-\frac{1}{2}}dx. \tag{14}$$

2. Preliminaries and statements of results

In this section, we will recall some elementary facts on orthogonal polynomials and state our main results in this paper. More details on orthogonal polynomials can be found either in the books [1,2,10] or in the papers [8,9].

The falling factorial sequence $(x)_n$ and the rising factorial sequence $\langle x\rangle_n$ are respectively given by

$$(x)_n = x(x-1)\cdots(x-n+1),\ (n\geq 1),\ (x)_0=1, \tag{15}$$

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1),\ (n\geq 1), \langle x\rangle_0=1. \tag{16}$$

The two factorials are related by

$$(-1)^n(x)_n=\langle -x\rangle_n,\quad (-1)^n\langle x\rangle_n=(-x)_n. \tag{17}$$

$$\frac{(2n-2s)!}{(n-s)!}=\frac{2^{2n-2s}(-1)^s\langle\frac{1}{2}\rangle_n}{\langle\frac{1}{2}-n\rangle_s},\quad (n\geq s\geq 0). \tag{18}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+1-n)}=(x)_n,\quad \frac{\Gamma(x+n)}{\Gamma(x)}=\langle x\rangle_n,\quad (n\geq 0), \tag{19}$$

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!\sqrt{\pi}}{2^{2n}n!},\quad (n\geq 0), \tag{20}$$

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\quad (\mathrm{Re}(x)>0,\ \mathrm{Re}(y)>0), \tag{21}$$

where $\Gamma(x)$ and $B(x,y)$ are the gamma and beta functions, respectively.

The hypergeometric function is defined by

$${}_pF_q(a_1,\ldots,a_p\ ;\ b_1,\ldots,b_q\ ;x)=\sum_{n=0}^{\infty}\frac{\langle a_1\rangle_n\cdots\langle a_p\rangle_n}{\langle b_1\rangle_n\cdots\langle b_q\rangle_n}\frac{x^n}{n!},\quad (p\leq q+1,\ |x|<1). \tag{22}$$

(23) In particular, the Gaussian hypergeometric function is given by

$${}_2F_1(a,b\ ;\ c\ ;\ x)=\sum_{n=0}^{\infty}\frac{\langle a\rangle_n\langle b\rangle_n}{\langle c\rangle_n}\frac{x^n}{n!},\quad (|x|<1). \tag{24}$$

Next, we will recall Chebyshev polynomials of the second kind $U_n(x)$, those of the third kind $V_n(x)$, those of the fourth kind $W_n(x)$, Hermite polynomials $H_n(x)$, generalized Laguerre polynomials $L_n^{\alpha}(x)$, Legendre polynomials $P_n(x)$, Gegenbauer polynomials $C_n^{(\lambda)}(x)$, and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$.

They are explicitly given by

$$
\begin{aligned}
(25)\qquad U_n(x) &= (n+1)\,{}_2F_1\Big(-n,\ n+2\ ;\ \frac{3}{2}\ ;\frac{1-x}{2}\Big) = \sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n-l}{l}(2x)^{n-2l},\ (n\geq 0),\\
(26)\qquad V_n(x) &= {}_2F_1\Big(-n,\ n+1\ ;\ \frac{1}{2}\ ;\frac{1-x}{2}\Big) = \sum_{l=0}^{n}\binom{2n-l}{l}2^{n-l}(x-1)^{n-l},\ (n\geq 0),\\
(27)\qquad W_n(x) &= (2n+1)\,{}_2F_1\Big(-n,\ n+1\ ;\ \frac{3}{2}\ ;\frac{1-x}{2}\Big)\\
&= (2n+1)\sum_{l=0}^{n}\frac{2^{n-l}}{2n-2l+1}\binom{2n-l}{l}(x-1)^{n-l},\ (n\geq 0),\\
(28)\qquad H_n(x) &= n!\sum_{l=0}^{[\frac{n}{2}]}\frac{(-1)^l}{l!(n-2l)!}(2x)^{n-2l},\quad (n\geq 0),\\
(29)\qquad L_n^{\alpha}(x) &= \frac{\langle\alpha+1\rangle_n}{n!}\,{}_1F_1(-n\ ;\ \alpha+1\ ;\ x) = \sum_{l=0}^{n}\frac{(-1)^l\binom{n+\alpha}{n-l}}{l!}x^l,\quad (n\geq 0),\\
(30)\qquad P_n(x) &= {}_2F_1\Big(-n,\ n+1,\ ;\ 1;\ \frac{1-x}{2}\Big) = \frac{1}{2^n}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\binom{n}{l}\binom{2n-2l}{n}x^{n-2l},\quad (n\geq 0),\\
(31)\qquad C_n^{(\lambda)}(x) &= \binom{n+2\lambda-1}{n}{}_2F_1\Big(-n,\ n+2\lambda\ ;\ \lambda+\frac{1}{2}\ ;\ \frac{1-x}{2}\Big)\\
&= \sum_{k=0}^{[\frac{n}{2}]}(-1)^k\frac{\Gamma(n-k+\lambda)}{\Gamma(\lambda)k!(n-2k)!}(2x)^{n-2k},\quad (n\geq 0),\\
(32)\qquad P_n^{(\alpha,\beta)}(x) &= \frac{\langle\alpha+1\rangle_n}{n!}\,{}_2F_1\Big(-n,\ 1+\alpha+\beta+n\ ;\ \alpha+1\ ;\ \frac{1-x}{2}\Big)\\
&= \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\binom{n+\beta}{k}\Big(\frac{x-1}{2}\Big)^k\Big(\frac{x+1}{2}\Big)^{n-k},\quad (n\geq 0),
\end{aligned}
$$

where $\lambda > -\frac{1}{2}$ and $\lambda \neq 0$.

In Theorem 1 of [8] and Theorem 1.1 of [9], the difference of sums of finite products of Chebyshev polynomials of the first kind

$$
\begin{aligned}
&\sum_{l=0}^{n}\sum_{i_1+\cdots+i_{r+1}=n-l}\binom{r+l}{l}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x)\\
&-\sum_{l=0}^{n-2}\sum_{i_1+\cdots+i_{r+1}=n-l-2}\binom{r+l}{l}x^l T_{i_1}(x)\cdots T_{i_{r+1}}(x),\quad (n\geq 2,\ r\geq 1),
\end{aligned}
$$

were expressed in terms of orthogonal polynomials in (8) and (25)-(32).

The next theorem is obtained from those expressions by replacing x by $3x$ and making use of (9).

Theorem 2.1. *Let n,r be integers with $n \geq 2,\ r \geq 1$. Then we have the following representations.*

$$\sum_{l=0}^{n} \sum_{i_1+\cdots+i_{r+1}=n-l} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x)$$
$$-\sum_{l=0}^{n-2} \sum_{i_1+\cdots+i_{r+1}=n-l-2} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x)$$

$$= \binom{n+r}{r} \sum_{j=0}^{[\frac{n}{2}]} \varepsilon_{n-2j} \binom{n}{j} {}_2F_1(-j,\ j-n\,;1-n-r\,;1) C_{n-2j}(x) \tag{33}$$

$$= \frac{1}{n+1} \binom{n+r}{r} \sum_{j=0}^{[\frac{n}{2}]} \binom{n+1}{j} (n-2j+1) {}_2F_1(-j,\ j-n-1\,;1-n-r\,;1) U_{n-2j}(3x) \tag{34}$$

$$= \binom{n+r}{r} \sum_{j=0}^{n} \binom{n}{[\frac{j}{2}]} {}_2F_1\Big(-\Big[\frac{j}{2}\Big],\ \Big[\frac{j}{2}\Big]-n\,;1-n-r\,;1\Big) V_{n-j}(3x) \tag{35}$$

$$= \binom{n+r}{r} \sum_{j=0}^{n} (-1)^j \binom{n}{[\frac{j}{2}]} {}_2F_1\Big(-\Big[\frac{j}{2}\Big],\ \Big[\frac{j}{2}\Big]-n\,;1-n-r\,;1\Big) W_{n-j}(3x) \tag{36}$$

$$= \frac{(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j!(n-2j)!} {}_1F_1(-j\ ;\ 1-n-r\ ;\ -1) H_{n-2j}(3x) \tag{37}$$

$$= \frac{(n+r)2^n}{r!} \sum_{j=0}^{n} \frac{(-1)^j}{\Gamma(\alpha+j+1)} \sum_{l=0}^{[\frac{n-j}{2}]} \frac{(-\frac{1}{4})^l (n+r-1-l)!\Gamma(n+\alpha+1-2l)}{l!(n-j-2l)!} L_j^{\alpha}(3x) \tag{38}$$

$$= \frac{(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(2n-4j+1)}{j!(n-j+\frac{1}{2})_{n-j}} {}_2F_1(-j,\ j-n-\frac{1}{2}\ ;\ 1-n-r\ ;\ 1) P_{n-2j}(3x) \tag{39}$$

$$= \frac{\Gamma(\lambda)(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n+\lambda-2j)}{j!\Gamma(n+\lambda+1-j)} {}_2F_1(-j,\ j-n-\lambda\ ;\ 1-n-r\ ;\ 1) C_{n-2j}^{(\lambda)}(3x) \tag{40}$$

$$= \frac{(-2)^n(n+r)!}{r!} \sum_{j=0}^{n} \frac{(-2)^j\Gamma(j+\alpha+\beta+1)}{\Gamma(2j+\alpha+\beta+1)} \sum_{l=0}^{[\frac{n-j}{2}]} \frac{(-\frac{1}{4})^l}{l!(n+r-1)_l(n-j-2l)!} \tag{41}$$
$$\times {}_2F_1(j+2l-n,\ j+\beta+1\,;2j+\alpha+\beta+2\,;2) P_j^{(\alpha,\beta)}(3x).$$

The next theorem will be shown in the following section. We note here that the polynomials on the right hand sides are in x, not in $3x$.

Theorem 2.2. *Let n,r be integers with $n \geq 2,\ r \geq 1$. Then we have the following representations*

$$\sum_{l=0}^{n} \sum_{i_1+\cdots+i_{r+1}=n-l} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x)$$
$$-\sum_{l=0}^{n-2} \sum_{i_1+\cdots+i_{r+1}=n-l-2} \binom{r+l}{r} (3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x)$$

$$(42) \quad = 3^n \binom{n+r}{r} \sum_{j=0}^{[\frac{n}{2}]} \varepsilon_{n-2j} \binom{n}{j} {}_2F_1\left(-j,\ j-n\,; 1-n-r\,; \frac{1}{9}\right) T_{n-2j}(x)$$

$$(43) \quad = \frac{3^n}{n+1} \binom{n+r}{r} \sum_{j=0}^{[\frac{n}{2}]} \binom{n+1}{j} (n-2j+1) {}_2F_1\left(-j,\ j-n-1\ ;\ 1-n-r\,; \frac{1}{9}\right) U_{n-2j}(x)$$

$$(44) \quad = 3^n \binom{n+r}{r} \sum_{j=0}^{n} \binom{n}{[\frac{j}{2}]} {}_2F_1\left(-\left[\frac{j}{2}\right],\ \left[\frac{j}{2}\right]-n\,; 1-n-r\ ;\ \frac{1}{9}\right) V_{n-j}(x)$$

$$(45) \quad = 3^n \binom{n+r}{r} \sum_{j=0}^{n} (-1)^j \binom{n}{[\frac{j}{2}]} {}_2F_1\left(-\left[\frac{j}{2}\right],\ \left[\frac{j}{2}\right]-n\,; 1-n-r\ ;\ \frac{1}{9}\right) W_{n-j}(x)$$

$$(46) \quad = \frac{3^n (n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{1}{j!(n-2j)!} {}_1F_1\left(-j\ ;\ 1-n-r\ ;\ -\frac{1}{9}\right) H_{n-2j}(x)$$

$$(47) \quad = \frac{6^n (n+r)}{r!} \sum_{j=0}^{n} \frac{(-1)^j}{\Gamma(\alpha+j+1)} \times \sum_{l=0}^{[\frac{n-j}{2}]} \frac{(-\frac{1}{36})^l (n+r-1-l)!\Gamma(n+\alpha+1-2l)}{l!(n-j-2l)!} L_j^{\alpha}(x)$$

$$(48) \quad = \frac{3^n (n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(2n-4j+1)}{j!(n-j+\frac{1}{2})_{n-j}} {}_2F_1\left(-j,\ j-n-\frac{1}{2}\ ;\ 1-n-r\ ;\ \frac{1}{9}\right) P_{n-2j}(x)$$

$$(49) \quad = \frac{3^n \Gamma(\lambda)(n+r)!}{r!} \sum_{j=0}^{[\frac{n}{2}]} \frac{(n+\lambda-2j)}{j!\Gamma(n+\lambda+1-j)} {}_2F_1\left(-j,\ j-n-\lambda\ ;\ 1-n-r\ ;\ \frac{1}{9}\right) C_{n-2j}^{(\lambda)}(x)$$

$$(50) \quad = \frac{(-6)^n (n+r)!}{r!} \sum_{j=0}^{n} \frac{(-2)^j \Gamma(j+\alpha+\beta+1)}{\Gamma(2j+\alpha+\beta+1)} \sum_{l=0}^{[\frac{n-j}{2}]} \frac{(-\frac{1}{36})^l}{l!(n+r-1)_l (n-j-2l)!}$$
$$\times {}_2F_1(j+2l-n,\ j+\beta+1\,; 2j+\alpha+\beta+2\,; 2) P_j^{(\alpha,\beta)}(x).$$

3. PROOF OF THEOREM 2.2

Here we are going to show Theorem 2.2. The next two propositions are respectively from Proposition 1 in [8] and Proposition 2.1 in [9].

Proposition 3.1. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then we have the following.*

(a)
$$q(x) = \sum_{k=0}^{n} C_{k,1} T_k(x),$$

where

$$C_{k,1} = \frac{(-1)^k 2^k k! \varepsilon_k}{(2k)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k-\frac{1}{2}} dx.$$

(b)
$$q(x) = \sum_{k=0}^{n} C_{k,2} U_k(x),$$

where

$$C_{k,2} = \frac{(-1)^k 2^{k+1} (k+1)!}{(2k+1)! \pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\frac{1}{2}} dx.$$

(c)
$$q(x) = \sum_{k=0}^{n} C_{k,3} V_k(x),$$

where

$$C_{k,3} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k-\frac{1}{2}} (1+x)^{k+\frac{1}{2}} dx.$$

(d)
$$q(x) = \sum_{k=0}^{n} C_{k,4} W_k(x),$$

where

$$C_{k,4} = \frac{(-1)^k k! 2^k}{(2k)!\pi} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\frac{1}{2}} (1+x)^{k-\frac{1}{2}} dx.$$

Proposition 3.2. *Let $q(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Then the following hold true.*

(a)
$$q(x) = \sum_{k=0}^{n} C_{k,5} H_k(x),$$

where

$$C_{k,5} = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} q(x) \frac{d^k}{dx^k} e^{-x^2} dx.$$

(b)
$$q(x) = \sum_{k=0}^{n} C_{k,6} L_k^{\alpha}(x),$$

where

$$C_{k,6} = \frac{1}{\Gamma(\alpha+k+1)} \int_{0}^{\infty} q(x) \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}) dx.$$

(c)
$$q(x) = \sum_{k=0}^{n} C_{k,7} P_k(x),$$

where

$$C_{k,7} = \frac{2k+1}{2^{k+1} k!} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (x^2-1)^k dx.$$

(d)
$$q(x) = \sum_{k=0}^{n} C_{k,8} C_k^{(\lambda)}(x),$$

where

$$C_{k,8} = \frac{(k+\lambda)\Gamma(\lambda)}{(-2)^k \sqrt{\pi}\, \Gamma(k+\lambda+\frac{1}{2})} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x^2)^{k+\lambda-\frac{1}{2}} dx.$$

(e)
$$q(x) = \sum_{k=0}^{n} C_{k,9} P_k^{(\alpha,\beta)}(x),$$

where

$$C_{k,9} = \frac{(-1)^k (2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{2^{k+\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)} \int_{-1}^{1} q(x) \frac{d^k}{dx^k} (1-x)^{k+\alpha} (1+x)^{k+\beta} dx.$$

We also need the next two results from Proposition 2 in [8] and Proposition 2.2 in [9].

Proposition 3.3. *Let m,k be nonnegative integers. Then we have the following.*

(a) $$\int_{-1}^{1}(1-x^2)^{k-\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(b) $$\int_{-1}^{1}(1-x^2)^{k+\frac{1}{2}}x^m dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k+2)!\pi}{2^{m+2k+2}(\frac{m}{2}+k+1)!(\frac{m}{2})!(k+1)!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(c) $$\int_{-1}^{1}(1-x)^{k-\frac{1}{2}}(1+x)^{k+\frac{1}{2}}x^m dx = \begin{cases} \frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(d) $$\int_{-1}^{1}(1-x)^{k+\frac{1}{2}}(1+x)^{k-\frac{1}{2}}x^m dx = \begin{cases} -\frac{(m+1)!(2k)!\pi}{2^{m+2k+1}(\frac{m+1}{2}+k)!(\frac{m+1}{2})!k!}, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!(2k)!\pi}{2^{m+2k}(\frac{m}{2}+k)!(\frac{m}{2})!k!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

Proposition 3.4. *Let m,k be nonnegative integers. Then we have the following.*

(a) $$\int_{-\infty}^{\infty}x^m e^{-x^2}dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{m!\sqrt{\pi}}{(\frac{m}{2})!2^m}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(b) $$\int_{-1}^{1}x^m(1-x^2)^k dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{2^{2k+2}k!m!(k+\frac{m}{2}+1)!}{(\frac{m}{2})!(2k+m+2)!}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(c) $$\int_{-1}^{1}x^m(1-x^2)^{k+\lambda-\frac{1}{2}}dx = \begin{cases} 0, & \text{if } m \equiv 1 \pmod 2, \\ \frac{\Gamma(k+\lambda+\frac{1}{2})\Gamma(\frac{m}{2}+\frac{1}{2})}{\Gamma(k+\lambda+\frac{m}{2}+1)}, & \text{if } m \equiv 0 \pmod 2. \end{cases}$$

(d) $$\int_{-1}^{1}x^m(1-x)^{k+\alpha}(1+x)^{k+\beta}dx = 2^{2k+\alpha+\beta+1}\sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}2^s\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}.$$

The following lemma is needed in showing all the identities in (42)-(50).

Lemma 3.5. *Let n,r be integers with $n \geq 2,\ r \geq 1$. Then the following identity holds.*

$$\sum_{l=0}^{n}\sum_{i_1+\cdots+i_{r+1}=n-l}\binom{r+l}{r}(3x)^l C_{i_1}(x)\cdots C_{i_{r+1}}(x) - \sum_{l=0}^{n-2}\sum_{i_1+\cdots+i_{r+1}=n-l-2}\binom{r+l}{r}(3x)^l C_{i_1}(x)\cdots C_{i_{r+1}(x)} = \frac{1}{3\cdot 6^{r-1}r!}C_{n+r}^{(r)}(x). \tag{51}$$

Proof. Let

$$F(t,x) = \frac{1-3xt}{1-6xt+t^2} = \sum_{n=0}^{\infty}C_n(x)t^n. \tag{52}$$

By taking the r-th derivative with respect to x on both sides of (52), we obtain

$$3(t-t^3)r!(6t)^{r-1}(1-6xt+t^2)^{-(r+1)} = \sum_{n=0}^{\infty}C_{n+r}^{(r)}(x)t^{n+r}, \quad (r \geq 1). \tag{53}$$

Then, from (53), we get

$$\begin{aligned}\frac{1}{3\cdot 6^{r-1}r!}&\sum_{n=0}^{\infty}C_{n+r}^{(r)}(x)t^n\\
&=\frac{1-t^2}{(1-3xt)^{r+1}}\left(\frac{1-3xt}{1-6xt+t^2}\right)^{r+1}\\
&=(1-t^2)\sum_{j=0}^{\infty}\binom{r+j}{j}(3xt)^j\sum_{l=0}^{\infty}\left(\sum_{i_1+\cdots+i_{r+1}=l}C_{i_1}(x)\cdots C_{i_{r+1}}(x)\right)t^l\\
&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{r+l}{r}(3x)^l\sum_{i_1+\cdots+i_{r+1}=n-l}C_{i_1}(x)\cdots C_{i_{r+1}}(x)\right)t^n\\
&\quad-\sum_{n=2}^{\infty}\left(\sum_{l=0}^{n-2}\binom{r+l}{r}(3x)^l\sum_{i_1+\cdots+i_{r+1}=n-l-2}C_{i_1}(x)\cdots C_{i_{r+1}}(x)\right)t^n,\end{aligned}$$

from which the desired result follows. □

Remark : In Lemma 1 of [8], it was mentioned that

$$\begin{aligned}&\sum_{l=0}^{n}\sum_{i_1+\cdots+i_{r+1}=n-l}\binom{r+l}{r}x^lT_{i_1}(x)\cdots T_{i_{r+1}}(x)\\
&-\sum_{l=0}^{n-2}\sum_{i_1+\cdots+i_{r+1}=n-l-2}\binom{r+l}{r}x^lT_{i_1}(x)\cdots T_{i_{r+1}}(x)=\frac{1}{2^{r-1}r!}T_{n+r}^{(r)}(x),\quad (n\geq 2,\ r\geq 1).\end{aligned}\tag{54}$$

We note here that (51) also follows from from (54) by replacing x by $3x$ and using (9).
For brevity and for $n\geq 2,\ r\geq 1$, we let

$$\begin{aligned}\alpha_{n,r}(x)&=\sum_{l=0}^{n}\sum_{i_1+\cdots+i_{r+1}=n-l}\binom{r+l}{r}(3x)^lC_{i_1}(x)\cdots C_{i_{r+1}}(x)\\
&\quad-\sum_{l=0}^{n-2}\sum_{i_1+\cdots+i_{r+1}=n-l-2}\binom{r+l}{r}(3x)^lC_{i_1}(x)\cdots C_{i_{r+1}}(x).\end{aligned}\tag{55}$$

From (8) and (9), we obtain an explicit expression for $C_n(x)$:

$$C_n(x)=\frac{n}{2}\sum_{l=0}^{[\frac{n}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}(6x)^{n-2l}.\tag{56}$$

Thus the r-th derivative of (56) is given by

$$C_n^{(r)}(x)=\frac{n}{2}\sum_{l=0}^{[\frac{n-r}{2}]}(-1)^l\frac{1}{n-l}\binom{n-l}{l}6^{n-2l}(n-2l)_rx^{n-2l-r}.\tag{57}$$

In particular, we see that

$$C_{n+r}^{(r+k)}(x)=\frac{n+r}{2}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\frac{1}{n+r-l}\binom{n+r-l}{l}6^{n+r-2l}(n+r-2l)_{r+k}x^{n-k-2l}.\tag{58}$$

Every expression in (42)-(50) can be derived from Propositions 3.1- 3.4, (51), (58), and the facts in (15)-(21). Here we are going to show only (44) and (50), while the rest are left as exercises to the interested reader.

Let

$$\alpha_{n,r}(x)=\sum_{k=0}^{n}C_{k,3}V_k(x). \tag{59}$$

Then, from (c) in Proposition 3.1, (51), (58), and integration by parts k times, we have

$$\begin{aligned}
C_{k,3}&=\frac{(-1)^k k!2^k}{(2k)!\pi}\int_{-1}^{1}\alpha_{n,r}(x)\frac{d^k}{dx^k}(1-x)^{k-\frac12}(1+x)^{k+\frac12}dx\\
&=\frac{(-1)^k k!2^k}{3(2k)!\pi 6^{r-1}r!}\int_{-1}^{1}C_{n+r}^{(r)}(x)\frac{d^k}{dx^k}(1-x)^{k-\frac12}(1+x)^{k+\frac12}dx\\
&=\frac{k!2^k}{3(2k)!\pi 6^{r-1}r!}\int_{-1}^{1}C_{n+r}^{(r+k)}(x)(1-x)^{k-\frac12}(1+x)^{k+\frac12}dx\\
&=\frac{k!2^k}{3(2k)!\pi 6^{r-1}r!}\frac{n+r}{2}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\frac{1}{n+r-l}\binom{n+r-l}{l}6^{n+r-2l}\\
&\quad\times(n+r-2l)_{r+k}\int_{-1}^{1}x^{n-k-2l}(1-x)^{k-\frac12}(1+x)^{k+\frac12}dx.
\end{aligned}\tag{60}$$

From (c) of Proposition 3.3, we note that

$$\begin{aligned}
&\int_{-1}^{1}x^{n-k-2l}(1-x)^{k-\frac12}(1+x)^{k+\frac12}dx\\
&=\begin{cases}\dfrac{(n-k-2l+1)!(2k)!\pi}{2^{n+k-2l+1}\left(\frac{n+k+1}{2}-l\right)!\left(\frac{n-k+1}{2}-l\right)!k!}, & \text{if } n\not\equiv k \pmod 2,\\[2ex] \dfrac{(n-k-2l)!(2k)!\pi}{2^{n+k-2l}\left(\frac{n+k}{2}-l\right)!\left(\frac{n-k}{2}-l\right)!k!}, & \text{if } n\equiv k \pmod 2.\end{cases}
\end{aligned}\tag{61}$$

Then, from (59)-(61) and after some simplifications, we get

$$\begin{aligned}
\alpha_{n,r}(x)&=\frac{3^n(n+r)!}{r!}\sum_{\substack{0\le k\le n\\ n\not\equiv k \pmod 2}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac19)^l(n+r-1-l)!}{l!(n+r-1)!}\frac{(n-k-2l+1)V_k(x)}{2\left(\frac{n+k+1}{2}-l\right)!\left(\frac{n-k+1}{2}-l\right)!}\\
&+\frac{3^n(n+r)!}{r!}\sum_{\substack{0\le k\le n\\ n\equiv k \pmod 2}}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac19)^l(n+r-1-l)!}{l!(n+r-1)!}\frac{V_k(x)}{\left(\frac{n+k}{2}-l\right)!\left(\frac{n-k}{2}-l\right)!}.
\end{aligned}\tag{62}$$

Letting in (62) $n-k=2j+1$, for the first sum and $n-k=2j$, for the second, and after some simplifications, we have

$$\begin{aligned}
\alpha_{n,r}(x)&=\frac{3^n(n+r)!}{r!}\sum_{j=0}^{[\frac{n-1}{2}]}\frac{1}{(n-j)!j!}{}_2F_1\Big(-j,\ j-n\,;1-n-r\,;\ \frac19\Big)V_{n-2j-1}(x)\\
&+\frac{3^n(n+r)!}{r!}\sum_{j=0}^{[\frac{n}{2}]}\frac{1}{(n-j)!j!}{}_2F_1\Big(-j,\ j-n\,;1-n-r\,;\ \frac19\Big)V_{n-2j}(x)\\
&=\frac{3^n(n+r)!}{r!}\sum_{j=0}^{n}\frac{1}{(n-[\frac{j}{2}])![\frac{j}{2}]!}{}_2F_1\Big(-\Big[\frac{j}{2}\Big],\ \Big[\frac{j}{2}\Big]-n\,;\ 1-n-r\,;\ \frac19\Big)V_{n-j}(x).
\end{aligned}\tag{63}$$

This completes the proof for (44).

Next, we let

$$\alpha_{n,r}(x)=\sum_{k=0}^{n}C_{k,9}P_k^{(\alpha,\beta)}(x). \tag{64}$$

Then, from (e) of Proposition 3.2, (51), (58), and integration by parts k times, and proceeding just as in (60), we obtain

$$\begin{aligned}C_{k,9} \;=\; &\frac{(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{3\cdot 2^{k+\alpha+\beta+1}\Gamma(\alpha+k+1)\Gamma(\beta+k+1)6^{r-1}r!}\\ &\times\frac{n+r}{2}\sum_{l=0}^{[\frac{n-k}{2}]}(-1)^l\frac{1}{n+r-l}\binom{n+r-l}{l}6^{n+r-2l}(n+r-2l)_{r+k}\\ &\times\int_{-1}^{1}x^{n-k-2l}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx.\end{aligned} \tag{65}$$

Here, from (d) of Proposition 3.4, we obsreve that

$$\begin{aligned}\int_{-1}^{1}x^{n-k-2l}(1-x)^{k+\alpha}(1+x)^{k+\beta}dx=2^{2k+\alpha+\beta+1}&\sum_{s=0}^{n-k-2l}\binom{n-k-2l}{s}(-1)^{n-k-2l-s}2^s\\ &\times\frac{\Gamma(k+\alpha+1)\Gamma(k+\beta+s+1)}{\Gamma(2k+\alpha+\beta+s+2)}.\end{aligned} \tag{66}$$

From (65) and (66), and after some simplifications, we have

$$\begin{aligned}C_{k,9} \;=\; &\frac{(-6)^n(n+r)!}{r!}\frac{(-2)^k(2k+\alpha+\beta+1)\Gamma(k+\alpha+\beta+1)}{\Gamma(\beta+k+1)}\\ &\times\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac{1}{36})^l}{l!(n+r-1)_l}\sum_{s=0}^{n-k-2l}\frac{(-2)^s\Gamma(k+\beta+s+1)}{s!(n-k-2l-s)!\Gamma(2k+\alpha+\beta+s+2)},\end{aligned} \tag{67}$$

where we note that the inner sum is equal to

$$\begin{aligned}&\sum_{s=0}^{n-k-2l}\frac{(-2)^s\Gamma(k+\beta+s+1)}{s!(n-k-2l-s)!\Gamma(2k+\alpha+\beta+s+2)}\\ &=\frac{\Gamma(k+\beta+1)}{\Gamma(2k+\alpha+\beta+2)(n-k-2l)!}{}_2F_1(k+2l-n,\ k+\beta+1\ ;\ 2k+\alpha+\beta+2;\ 2).\end{aligned} \tag{68}$$

Now, form (64), (67), and (68), we finally get

$$\begin{aligned}\alpha_{n,r}(x)=&\frac{(-6)^n(n+r)!}{r!}\sum_{k=0}^{n}\frac{(-2)^k\Gamma(k+\alpha+\beta+1)}{\Gamma(2k+\alpha+\beta+1)}\sum_{l=0}^{[\frac{n-k}{2}]}\frac{(-\frac{1}{36})^l}{l!(n+r-1)_l(n-k-2l)!}\\ &\times{}_2F_1(k+2l-n,\ k+\beta+1\ ;\ 2k+\alpha+\beta+2\ ;\ 2)P_k^{(\alpha,\beta)}(x).\end{aligned}$$

This shows the desired result in (50).

4. Conclusions

Behera and Panda introduced balancing numbers about twenty years ago. Since then, these numbers have been intensively studied by many researchers and lots of interesting properties of them have been unveiled. Natural extensions of the Lucas-balancing numbers are the Lucas-balancing polynomials. In this paper, we consider a difference of sums of finite products of Lucas-balancing polynomials and represent them in terms of nine orthogonal polynomials in two different ways each. In particular, this gives us an expression of such a difference of sums of finite products in terms of Lucas-balancing polynomials. Our proof is based on the recent observation by Frontczak as to a fundamental relation between Chebyshev polynomials of the first kind and Lucas-balancing polynomials.

5. acknowledgements

This work was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) (No. 2017R1A2B4006092).

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